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ON PANKING THE PLAYERS IN A 3-PLAYER TOURNAMENT

Robert E. Bechhofer

Cornell University
Ithaca, New York

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CORNELL UNIVERSITY
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by

Robert E. Bechhofer

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Robert E. Bechhofer

Department of Operations Research, Cornell University

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1. Introduction

The present paper is an outgrowth of comments made by the writer as an official discussant of the stimulating paper [4] by Milton Sobel and George Weiss presented at the Nonparametric Conference.[1/] In those sections of their paper in which 3-player tournaments are considered, Sobel and Weiss formulate their problem in such a way that only one of three decisions is permissible after the tournament is completed, i.e., one must choose one of the decisions "Player 1 is best," "Player 2 is best," "Player 3 is best." As the writer pointed out at the time, this formulation of the problem may perhaps be a reasonable one in some situations (e.g., when the tournament director is told that a prize must be awarded to one of the players); but it would not be appropriate if one permits the decision that there is no clear-cut "best" player in the tournament -- to cope with those situations for which the possibility exists that (say) Player 1 is better than Player 2, Player 2 is better than Player 3, and Player 3 is better than Player 1. (The failure to provide for this contingency causes some of the Sobel-Weiss sequential procedures (those described in their Section 6, and which are based on the likelihood ratio statistic) to require an arbitrarily large number of observations to terminate sampling, when the true state of affairs is such that the probabilities that i beats j, that j beats k, and that k beats i ($i \neq j$, $j \neq k$, $k \neq i$, $i,j,k=1,2,3$) all approach one.) The formulation which we propose in the present paper (and which seems to us to be a very natural one) permits a fourth decision "no player is best," and is thus capable of dealing with such a possibility.

The present paper has two main parts. In the first part (Section 2) we formulate our ranking problem in much the same way as Sobel-Weiss [4]

1/ First International Symposium on Nonparametric Techniques in Statistical Inference held at Indiana University, Bloomington, Indiana, June 1-6, 1969.

do theirs, except that we add the possibility of a fourth decision to their three decisions. We add a probability requirement to their probability requirement, and propose a single-stage procedure which will guarantee both requirements. A numerical example is provided. The indifference-zone approach employed in [4] and in our Section 2 is similar in spirit to the approach used earlier by the authors ([1], [2]) for certain classes of ranking problems, although the present problem is quite different in structure from the ones which we considered earlier.

In the second part (Section 3) we formulate an identification problem which also parallels the one formulated by Sobel and Weiss, except that (as with the ranking problem) we add the possibility of a fourth decision, and an additional probability requirement. We propose a sequential identification procedure which will guarantee both requirements. Incorporated into our procedure is a certain prior probability distribution which is assumed to be known. The theory underlying the sequential identification procedures in both the Sobel-Weiss paper and the present paper is developed completely in [2] although some minor modifications are necessary to insure that our procedure guarantees our two requirements.

Our main interest really centers in the formulation of the ranking problem of Section 2, and in sequential procedures which guarantee the associated probability requirement. Our identification problem of Section 3, and the sequential procedure which guarantees the associated probability requirement can be regarded as a first step toward finding a sequential procedure for the ranking problem. The relation between such identification and ranking problems is discussed in Section 4. Finally, a more detailed consideration of certain aspects of the relation between the Sobel-Weiss paper and our paper is given in Section 5.

Sobel and Weiss proposed several intuitively appealing sampling rules for their sequential identification procedure. The Monte Carlo sampling results which they obtained using these rules are very striking, and demonstrate that this is a very fruitful area for future research. Their sampling rules can be incorporated in our sequential identification procedure, and we would anticipate that the performance of our procedure would thereby be improved.

2. A ranking problem

Let $p_{ij} = P\{\text{Player } i \text{ beats Player } j\}$ $(i \neq j;\ i,j = 1,2,3)$, $(0 \leq p_{ij} \leq 1)$; the p_{ij} are unknown, and it is also assumed that we have no knowledge as to whether $p_{ij} < 1/2$ or $p_{ij} > 1/2$. (We assume that the possibility $p_{ij} = 1/2$ cannot occur.[1/]) If $p_{ij} > 1/2$ we say that Player i is better than Player j. Let $\omega_{ab,cd,ef}$ be the State of Nature in which $p_{ab} > 1/2$, $p_{cd} > 1/2$, $p_{ef} > 1/2$. There are then 8 possible States of Nature $\omega_{12,23,31}, \ldots, \omega_{21,32,31}$ which we group into sets $\Omega_0, \ldots, \Omega_3$ as in Table I; the interpretation of each Ω_i is also given in the table.

Table I

State of Nature		Relation of p_{ij} to 1/2			Interpretation of Ω_i
		p_{12}	p_{23}	p_{13}	
Ω_0	$\omega_{12,23,31}$	>	>	<	No player is best
	$\omega_{21,32,13}$	<	<	>	
Ω_1	$\omega_{12,23,13}$	>	>	>	Player 1 is best
	$\omega_{12,32,13}$	>	<	>	
Ω_2	$\omega_{21,23,13}$	<	>	>	Player 2 is best
	$\omega_{21,23,31}$	<	>	<	
Ω_3	$\omega_{12,32,31}$	>	<	<	Player 3 is best
	$\omega_{21,32,31}$	<	<	<	

1/ $p_{ij} = 1/2$ represents the boundary regions between the Ω_i-regions given in Table I.

2.1 Goal, and probability requirement

The goal (purpose of the experiment) is stated below along with an associated probability requirement (R).

Goal: We wish to determine which one of the Ω_i $(i=0,1,2,3)$ represents the true State of Nature.

We permit four[1/] possible terminal decisions. These are given, along with their interpretation, in Table II.

Table II

Decision	Meaning of decision		
d_0	No player is best:	Either	$p_{12} > 1/2,\ p_{23} > 1/2,\ p_{13} < 1/2,$
		or	$p_{12} < 1/2,\ p_{23} < 1/2,\ p_{13} > 1/2.$
d_1	Player 1 is best:	Either	$p_{12} > 1/2,\ p_{23} > 1/2,\ p_{13} > 1/2,$
		or	$p_{12} > 1/2,\ p_{23} < 1/2,\ p_{13} > 1/2.$
d_2	Player 2 is best:	Either	$p_{12} < 1/2,\ p_{23} > 1/2,\ p_{13} > 1/2,$
		or	$p_{12} < 1/2,\ p_{23} > 1/2,\ p_{13} < 1/2.$
d_3	Player 3 is best:	Either	$p_{12} > 1/2,\ p_{23} < 1/2,\ p_{13} < 1/2,$
		or	$p_{12} < 1/2,\ p_{23} < 1/2,\ p_{13} < 1/2.$

1/ It is also possible to consider an alternative goal which is "To determine which one of the 8 ω's represents the true State of Nature." There would then be 8 possible terminal decisions. Corresponding changes would have to be made in the probability requirement (P).

Prior to the start of play we specify four constants $\{p_0, P_0^*; p_1, P_1^*\}$ with $1/2 < p_0, p_1 < 1$; $1/4 < P_0^*, P_1^* < 1$ which are incorporated into the following probability requirement.

Probability requirement (R):

We require that our procedure guarantee that

$$P_0 = P\{\text{Making decision } d_0\} \geq P_0^*$$

(2.1) R_0: when $\min\{p_{12}, p_{23}, p_{31}\} \geq p_0$

or

when $\min\{p_{21}, p_{32}, p_{13}\} \geq p_0$,

and

$$P_i = P\{\text{Making decision } d_i\} \geq P_1^* \qquad (i=1,2,3)$$

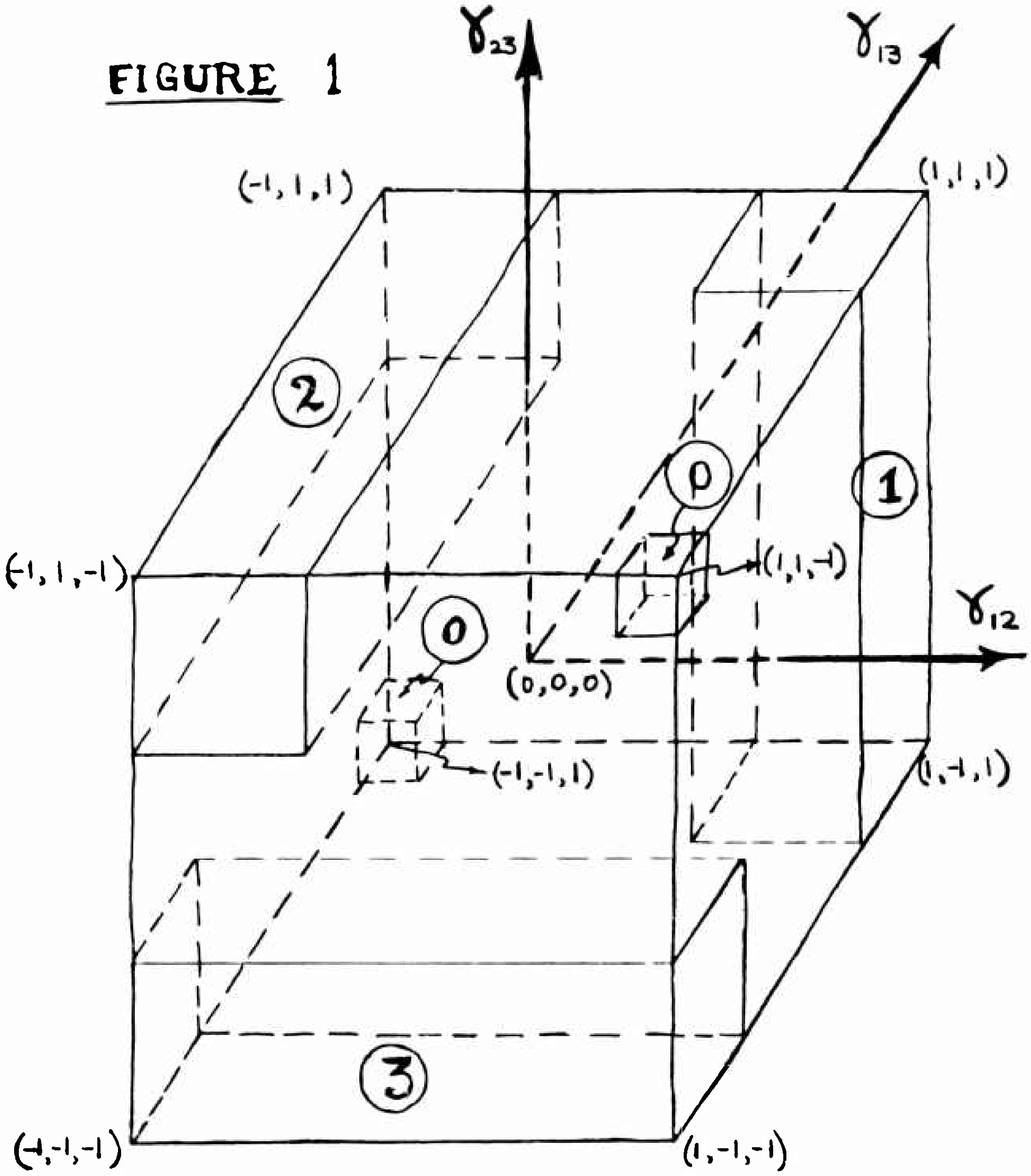
(2.2) R_1: when $\min\{p_{ij}, p_{ik}\} \geq p_1$

$(j \neq i; k \neq i; j \neq k; j,k=1,2,3)$.

2.2 Regions of preference and indifference

To assist the reader in visualizing the probability requirement, we give a geometric interpretation of the regions in the parameter space where particular decisions are preferred, and the region of indifference. The true State of Nature can be represented as a point $p = (p_{12}, p_{23}, p_{13})$ in the unit cube $(0 \leq p_{ij} \leq 1)$, $(i < j;\ i,j = 1,2,3)$. For simplicity of representation we let $\gamma_{ij} = 2(p_{ij} - 1/2)$, and transform the parameter space into the cube $(-1 \leq \gamma_{ij} \leq 1)$, $(i < j;\ i,j = 1,2,3)$ which is depicted in Figure 1.

FIGURE 1



Parameter Space: set of points $(\gamma_{12}, \gamma_{23}, \gamma_{13})$

with $-1 \leqq \gamma_{12}, \gamma_{23}, \gamma_{13} \leqq 1$

(i) = Region where decision d_i $(i = 0, 1, 2, 3)$ is preferred.

Then (i) in Figure 1 is the region in the parameter space of the γ_{ij} where decision d_i $(i = 0,1,2,3)$ is preferred. The region not included in the region of preference for any d_i is referred to as the region of indifference.

2.3 Single-stage procedure

There may be many statistical procedures (single-stage, two-stage, sequential) which will guarantee the probability requirement (R) of Section 2.1. We describe here a simple single-stage one.

SINGLE-STAGE RANKING PROCEDURE

Conduct a c-cycle tournament, a cycle consisting of 3 games {Player 1 vs. Player 2, Player 2 vs. Player 3, Player 1 vs. Player 3} where c is a predetermined number, computed as described in Section 2.4. Calculate $\hat{p}_{12}, \hat{p}_{23}, \hat{p}_{13}$ where $\hat{p}_{ij}$ = (proportion of games in which Player i beat Player j in the c-cycle tournament). Compare each of these $\hat{p}_{ij}$ to 1/2, and make the obvious terminal decision, e.g., make decision d_0 if $\hat{p}_{12} > 1/2$, $\hat{p}_{23} > 1/2$, $\hat{p}_{13} < 1/2$ or if $\hat{p}_{12} < 1/2$, $\hat{p}_{23} < 1/2$, $\hat{p}_{13} > 1/2$. Randomize between decisions if ties occur, e.g., randomize between decisions d_0 and d_2 with probability one-half if $\hat{p}_{12} = 1/2$, $\hat{p}_{23} > 1/2$, $\hat{p}_{13} < 1/2$, ties cannot occur if c is odd.

2.4 Determination of number of cycles $(\hat{c})$

Our problem now is to determine the smallest value of c that will guarantee R. For simplicity we consider only odd c (and thereby avoid the necessity of discussing ties). Denoting the cumulative binomial probability

$\Pr\{X \geqq x \mid n,p\} = \sum_{y=x}^{n} \frac{n!}{y!(n-y)!} p^y (1-p)^{n-y}$ by $B(x|n,p)$, we have that

$$(2.3) \qquad P_0 = B(\tfrac{c+1}{2}|c,p_{12}) \cdot B(\tfrac{c+1}{2}|c,p_{23}) \cdot B(\tfrac{c+1}{2}|c,p_{31})$$

$$+ [1 - B(\tfrac{c+1}{2}|c,p_{12})][1 - B(\tfrac{c+1}{2}|c,p_{23})][1 - B(\tfrac{c+1}{2}|c,p_{31})] ,$$

and for $j \neq i$, $k \neq i$, $j \neq k$; $j,k=1,2,3$ we have

$$(2.4) \qquad P_i = B(\tfrac{c+1}{2}|c,p_{ij}) \cdot B(\tfrac{c+1}{2}|c,p_{ik}) \qquad (i=1,2,3).$$

Since $p_0 > 1/2$ we have that P_0 is minimized, subject to $\min\{p_{12}, p_{23}, p_{31}\} \geqq p_0$, when $p_{12} = p_{23} = p_{31} = p_0$, or subject to $\max\{p_{12}, p_{23}, p_{31}\} \leqq 1 - p_0$ (which is equivalent to $\min\{p_{21}, p_{32}, p_{13}\} \geqq p_0$) when $p_{12} = p_{23} = p_{31} = 1 - p_0$; the same value of P_0 is attained when $p_{12} = p_{23} = p_{31} = p_0$ as when $p_{12} = p_{23} = p_{31} = 1 - p_0$. Also, P_i $(i=1,2,3)$ is minimized, subject to $\min\{p_{ij}, p_{ik}\} \geqq p_1$, when $p_{ij} = p_{ik} = p_1$. We refer to the configurations $p_{12} = p_{23} = p_{31} = p_0$ and $p_{12} = p_{23} = p_{31} = 1 - p_0$ as <u>least favorable</u> (LF) for R_0 when $\min\{p_{12}, p_{23}, p_{31}\} \geqq p_0$ or when $\max\{p_{12}, p_{23}, p_{31}\} \leqq 1 - p_0$, respectively, and the configurations $p_{ij} = p_{ik} = p_1$ as LF for R_1 when $\min\{p_{ij}, p_{ik}\} \geqq p_1$. Thus if we choose c_0 and c_1 as the smallest (odd) integers c which satisfy

$$(2.5) \qquad B^3(\tfrac{c+1}{2}|c,p_0) + [1 - B(\tfrac{c+1}{2}|c,p_0)]^3 \geqq P_0^*$$

and

(2.6) $$B^2\left(\frac{c+1}{2}\middle|c,p_1\right) \geq P_1^* ,$$

respectively, we see that

(2.7) $$\hat{c} = \max\{c_0, c_1\}$$

is the smallest (odd) integer which guarantees R.

2.5 Numerical example

Suppose that the experimenter sets up the specification $p_0 = 0.60$, $P_0^* = 0.75$; $p_1 = 0.65$, $P_1^* = 0.90$. We find using [4] that $[B(22|43,0.60)]^3 + [1 - B(22|43,0.60)]^3 = (0.908676)^3 + (0.091324)^3 = 0.751048$ while $[B(21|41,0.60)]^3 + [1 - B(21|41,0.60)]^3 = (0.903483)^3 + (0.096517)^3 = 0.73840$; hence, $c_0 = 43$. Also, $[B(15|29,0.65)]^2 = (0.952363)^2 = 0.90700$ while $[B(14|27,0.65)]^2 = (0.946377)^2 = 0.89563$; hence $c_1 = 29$. Thus $\hat{c} = \max\{43,29\} = 43$. But when $c = 43$ we also obtain $[B(22|43,0.65)]^2 = (0.978598)^2 = 0.95765$, $[B(22|43,0.64)]^2 = (0.970264)^2 = 0.94141$, $[B(22|43,0.63)]^2 = (0.959510)^2 = 0.92066$, $[B(22|43,0.62)]^2 = (0.945923)^2 = 0.89477$. Thus, for no extra cost in terms of sampling, i.e., using $\hat{c} = 43$, the experimenter could have set up and guaranteed stricter probability requirements R with specifications such as $\{p_0, P_0^*; p_1, P_1^*\} = \{0.60, 0.75; 0.65, 0.95765\}, \ldots,$ $\{0.60, 0.75; 0.63, 0.92066\}$.

3. An identification problem

Let p_{ij} be defined as in Section 2. We assume that p_0, p_1, θ are three given probabilities with $1/2 < p_0, p_1, \theta \leq 1$, and that there are 8 possible States of Nature $\omega^o_{12,23,31}, \ldots, \omega^o_{21,32,31}$ which represent certain possible pairings of the triple (p_{12}, p_{23}, p_{31}) with the probabilities p_0, p_1, θ. These pairings are given in Table III, as are the groupings of the ω^o's into sets $\Omega^o_0, \ldots, \Omega^o_3$. We further assume a known prior probability ξ, $\xi(\omega^o) \geq 0$ $(\sum_{\Omega^o} \xi(\omega^o) = 1)$ associated with each one of the 8 States of Nature in $\Omega^o = \bigcup_{i=0}^{3} \Omega^o_i$. For our problem it seems reasonable because of the symmetries, to set $\xi(\omega^o) = (1-3\lambda)/2$ for each of the 2 ω^o's in Ω^o_0, and $\xi(\omega^o) = \lambda/2$ for each of the 6 ω^o's in $\Omega^o_1 \cup \Omega^o_2 \cup \Omega^o_3$. Then $\xi_0 = \xi(\Omega^o_0) = 1-3\lambda$ and $\xi_i = \xi(\Omega^o_i) = \lambda$ $(i = 1,2,3)$.

Table III

State of Nature		Pairings of (p_{12}, p_{23}, p_{31}) with p_0, p_1, θ		
		p_{12}	p_{23}	p_{31}
Ω_0^o	$\omega^o_{12,23,31}$	p_0	p_0	p_0
	$\omega^o_{21,32,13}$	$1 - p_0$	$1 - p_0$	$1 - p_0$
Ω_1^o	$\omega^o_{12,23,13}$	p_1	θ	$1 - p_1$
	$\omega^o_{12,32,13}$	p_1	$1 - \theta$	$1 - p_1$
Ω_2^o	$\omega^o_{21,23,31}$	$1 - p_1$	p_1	θ
	$\omega^o_{21,23,13}$	$1 - p_1$	p_1	$1 - \theta$
Ω_3^o	$\omega_{12,32,31}$	θ	$1 - p_1$	p_1
	$\omega^o_{21,32,31}$	$1 - \theta$	$1 - p_1$	p_1

3.1 Goal, and probability requirement

The goal is stated below along with an associated probability requirement (R^o).

Goal: We wish to determine which one of the Ω_i^o $(i = 0,1,2,3)$ represents the true State of Nature.

We permit four possible terminal decisions. These are given, along with their interpretation, in Table IV.

Table IV

Decision	Meaning of decision
d_0	No player is best : Either $p_{12} = p_{23} = p_{31} = p_0$, or $p_{12} = p_{23} = p_{31} = 1 - p_0$.
d_1	Player 1 is best : Either $p_{12} = p_{13} = p_1$, $p_{23} = \theta$, or $p_{12} = p_{13} = p_1$, $p_{23} = 1 - \theta$.
d_2	Player 2 is best : Either $p_{21} = p_{23} = p_1$, $p_{31} = \theta$, or $p_{21} = p_{23} = p_1$, $p_{31} = 1 - \theta$.
d_3	Player 3 is best : Either $p_{32} = p_{31} = p_1$, $p_{12} = \theta$, or $p_{32} = p_{31} = p_1$, $p_{12} = 1 - \theta$.

Prior to the start of play we specify two constants $\{P_0^*, P_1^*\}$ with $1/4 < P_0^*, P_1^* < 1$ which are incorporated into the following probability requirement.

Probability requirement (R^o):

We require that our procedure guarantee that

(3.1) R_0^o : $P_0^o = P\{\text{Making decision } d_0 \text{ when } \Omega_0^o \text{ is true}\} \geq P_0^*$

and

(3.2) R_i^o : $P_i^o = P\{\text{Making decision } d_i \text{ when } \Omega_i^o \text{ is true}\} \geq P_1^*$.

$(i = 1,2,3)$

(The notation of (3.1) and (3.2) is justified by the fact that for simplicity we shall consider only procedures whose symmetry insures that for each i the probability expressions of (3.1) and (3.2) are the same for both members of Ω_i^0 $(i = 0,1,2,3)$.) It should be noted that in this problem we are deciding among four composite hypotheses (each one consisting of two completely specified States of Nature) with specified lower bounds on the probability of a correct decision for each of these States of Nature.

3.2 Sequential procedure

In this section we shall describe a sequential identification procedure which will guarantee the probability requirement (R^0). The theory underlying this development is derived in [2].

Our sequential procedure is based on 8 likelihoods, each one of which is associated with a different one of the eight States of Nature of Table III. These are defined in (3.3), below. In making these definitions we assume that the tournament is run in cycles (as defined in Section 2.3). However, this assumption of cyclic play is made only for convenience, and the same results would hold for more general symmetrical playing rules. Let $s_m^{(ij)} = s^{ij}$ (say) denote the number of "successes" for Player i when he plays Player j, i.e., the number of times that i beats j, in their first m games. We then define the 8 likelihoods

$$L_m^{(12,23,31)} = p_0^{s^{12}} (1-p_0)^{s^{21}} p_0^{s^{23}} (1-p_0)^{s^{32}} p_0^{s^{31}} (1-p_0)^{s^{13}}$$

$$= p_0^{s^{12}+s^{23}+s^{31}} (1-p_0)^{s^{21}+s^{32}+s^{13}}$$

$$(3.3) \qquad L_m^{(21,32,13)} = (1-p_0)^{s^{12}} p_0^{s^{21}} (1-p_0)^{s^{23}} p_0^{s^{32}} (1-p_0)^{s^{31}} p_0^{s^{13}}$$

$$= p_0^{s^{21}+s^{32}+s^{13}} (1-p_0)^{s^{12}+s^{23}+s^{31}}$$

$$L_m^{(12,23,13)} = p_1^{s^{12}} (1-p_1)^{s^{21}} \theta^{s^{23}} (1-\theta)^{s^{32}} (1-p_1)^{s^{31}} p_1^{s^{13}}$$

$$L_m^{(12,32,13)} = p_1^{s^{12}} (1-p_1)^{s^{21}} (1-\theta)^{s^{23}} \theta^{s^{32}} (1-p_1)^{s^{31}} p_1^{s^{13}}$$

etc.,

and the 4 likelihood sums

$$(3.4) \quad \begin{aligned} \mathcal{L}_m(d_0) &= L_m^{(12,23,31)} + L_m^{(21,32,13)} \\ \mathcal{L}_m(d_1) &= L_m^{(12,23,13)} + L_m^{(12,32,13)} \\ \mathcal{L}_m(d_2) &= L_m^{(21,23,31)} + L_m^{(21,23,13)} \\ \mathcal{L}_m(d_3) &= L_m^{(12,32,31)} + L_m^{(21,32,31)} \end{aligned} .$$

The à posteriori probability after cycle m that d_j is the correct decision given ξ and $s_m = (s_m^{(12)}, s_m^{(23)}, s_m^{(31)})$ is

$$(3.5) \quad P_{jm} = \frac{\xi_j \mathcal{L}_m(d_j)}{\sum_{k=0}^{3} \xi_k \mathcal{L}_m(d_k)} \qquad (j = 0,1,2,3).$$

The stopping and terminal decision rules of our sequential procedure are based on the P_{jm} of (3.5) and on a constant $\hat{P}^*(1/4 < \hat{P}^* < 1)$ which is chosen to guarantee (3.1) and (3.2). As stated in the preliminaries of Section 3, we shall let $\xi_0 = 1 - 3\lambda$ and $\xi_i = \lambda$ $(i = 1,2,3)$ where λ is assumed to be known.

SEQUENTIAL IDENTIFICATION PROCEDURE

Sampling rule: Run the tournament in cycles.

Stopping rule: Stop the tournament at the first cycle n for which

$$\max_{0 \leq j \leq 3} P^{o}_{jm} \geq \hat{P}^* \tag{3.6}$$

where

$$\hat{P}^* = \begin{cases} P^*_0 & \text{for} \quad \lambda = 0 \\ \max\{P^*_0 + 3\lambda(1-P^*_0),\ 1 - 3\lambda(1-P^*_1)\} & \text{for} \quad 0 < \lambda < 1/3 \\ P^*_1 & \text{for} \quad \lambda = 1/3. \end{cases} \tag{3.7}$$

Terminal decision rule: Make the decision which has associated with it the largest of the P^{o}_{jn}. If, because of equalities among the P^{o}_{jn}, several decisions are associated with the same maximum value, then select one such decision by a random device which assigns the same probability to each.

The terminal decision rule can be stated equivalently as follows. Letting $\bar{d}_j$ be the decision d_j for which $L_n(\bar{d}_j) = \max_{1 \leq j \leq 3} L_n(d_j)$, we:

$$\text{Choose } d_0 \text{ if } L_n(d_0) > \frac{\lambda}{1-3\lambda} L_n(\bar{d}_j) .$$

$$\text{Choose } \bar{d}_j \text{ if } L_n(d_0) < \frac{\lambda}{1-3\lambda} L_n(\bar{d}_j) . \tag{3.8}$$

Randomize among ties.

In the next section we show why $\hat{P}^*$ of (3.7) guarantees (3.1) and (3.2).

3.3 Determination of stopping constant (P^*)

We first remark that in general it is difficult to determine stopping constants which not only guarantee probability requirements such as R^c, but also do so in some optimal sense. This is particularly the case for nonsymmetric situations when the number of terminal decisions which the statistician is permitted to make is greater than two (which is our situation). In fact almost all of the theory contained in [2] which deals with sequential identification procedures is concerned with symmetric situations. However, it is possible to make use of certain results and remarks in [2] (specifically those in Section 3.7(c), pp. 51-52) to obtain solutions for certain particular problems involving nonsymmetric situations. No claim is made that these are the best possible solutions (which they certainly are not), but only that these solutions guarantee (3.1) and (3.2).

It follows from results proved in [2] (specifically from Lemma 3.1.1 and Corollary 3.1.1 -- see also equation (3.7.2)) that our sequential identification procedure of Section 3.2 using $P^*(1/4 < P^* < 1)$ on the r.h.s. of (3.6) guarantees

$$(3.8) \qquad 3\lambda P_1^o + (1 - 3\lambda)P_0^o \geq P^*$$

where P_0^o and $P_1^o = P_2^o = P_3^o$ are defined in (3.1) and (3.2), respectively. From (3.8) it follows, by setting first P_1^o and then P_0^o equal to one, that

$$(3.9) \qquad P_0^o \geq (P^* - 3\lambda)/(1 - 3\lambda) \qquad \text{for} \qquad 0 \leq \lambda < 1/3$$

(3.10) $\quad P_1^o \geq (P^* - 1 + 3\lambda)/3\lambda \quad$ for $\quad 0 < \lambda \leq 1/3.$

Since for fixed λ the r.h.s. of both (3.9) and (3.10) are increasing functions of P^* which approach one as P^* approaches one, it follows that we can choose P^* sufficiently large that the r.h.s. of (3.9) is $\geq P_0^*$ and the r.h.s. of (3.10) is $\geq P_1^*$, if we let $\hat{P}^*$ denote the smallest value of P^* which accomplishes this dual objective, we see that $\hat{P}^*$ is given by (3.7).

3.4 Numerical example

In Table V we give the values of $\hat{P}^*$ associated with the specification $P_0^* = 0.75$, $P_1^* = 0.90$ for selected values of $\xi_i = \lambda$ $(i = 1,2,3)$.

Table V

λ	$\hat{P}^*$	$\frac{\hat{P}^* - 3\lambda}{1-3\lambda}$	$\frac{\hat{P}^* - 1 + 3\lambda}{3\lambda}$
1/3	0.90	---	0.90
0.33	0.9975	0.75	0.9975
0.30	0.975	0.75	0.9722
5/21	13/14	0.75	0.90
0.20	0.94	0.85	0.90
0.10	0.97	0.9571	0.90
0.01	0.997	0.9969	0.90
0	0.75	0.75	---

We note that when $\hat{P}^*$ is considered as a function of λ for fixed $\{P^*_0, P^*_1\}$, it is discontinuous at $\lambda = 0$ and $\lambda = 1/3$, e.g., $\hat{P}^* \to 1$ as $\lambda \to 1/3$, but $\hat{P}^* = 0.90$ when $\lambda = 1/3$. However, this is to be expected: As $\lambda \to 1/3$, the à priori probability ξ_0 associated with Ω^o_0 approaches zero. But the procedure is still required to make the decision d_0 at least 75 percent of the time when Ω^o_0 occurs (however infrequently), as well as to make the decision d_i at least 90 percent of the time when Ω^o_i occurs $(i = 1,2,3)$. This will in general require many cycles to terminate the tournament (and hence the expected number of cycles to terminate the experiment will be very large). But when $\lambda = 1/3$, Ω^o_0 cannot occur, and the procedure is only required to decide among Ω^o_1, Ω^o_2, Ω^o_3 and make the decision d_i at least 90 percent of the time when Ω^o_i occurs $(i = 1,2,3)$. A similar analysis holds for $\lambda \to 0$ and $\lambda = 0$.

In Table V the minimum value of P^*, namely $13/14 = 0.9286$, occurs when $\lambda = 5/21$. In general this minimum value occurs when $\lambda = \bar{\lambda} = (1-P^*_0)/3(2-P^*_0-P^*_1)$ and is associated with $P^*_0 = (\hat{P}^* - 3\lambda)/(1-3\lambda)$ and $P^*_1 = (\hat{P}^* - 1 + 3\lambda)/3\lambda$.

We note that for $\lambda = 0.33$ (say), the experimenter could set up the stricter specification $\{P^*_0 = 0.75, P^*_1 = 0.9975\}$, and he can guarantee it with the same amount of sampling as is required by the present specification; such a tightening of the specification is possible for all λ-values $(0 < \lambda < 1/3)$ except $\lambda = 5/21$.

A non-Bayesian might regard λ $(0 \leq \lambda \leq 1/3)$ as a constant, at his disposal, to be set in such a way as to cause the procedure to have certain properties which he deems desirable. Thus he might <u>set</u> $\lambda = \bar{\lambda}$, and still be assured that the procedure will guarantee (3.1) and (3.2).

4. Relation between the ranking and identification probl

It was demonstrated in [2] that in many situations of practical interest, appropriate sequential procedures are more efficient than single-stage procedures which guarantee the same probability requirements, i.e., the sequential procedures often guarantee these requirements with a smaller expected number of observations. With this fact in mind, we would like for the problem of Section 2 to find a sequential ranking procedure, to replace the single-stage ranking procedure of that section, which will guarantee the probability requirement (2.1) and (2.2). As a first step in this direction, we set up an artificial identification problem, and displayed in Section 3.2 a sequential identification procedure which will guarantee the associated probability requirement (3.1) and (3.2) (which corresponds to protecting against the least favorable configuration of (2.1) and (2.2) with the addition of the nuisance parameter θ). (Usually, if one cannot solve the identification problem, there is little hope of solving the corresponding ranking problem.) The next logical step in the research process would be to try to produce a sequential ranking procedure which reduces to the sequential identification procedure when the parameters are in the least favorable configuration, and which for an arbitrary configuration of the parameters guarantees the probability requirement (2.1) and (2.2). (This is the same general approach as was used in [2]. See, in particular, Sections 3.6 and 6.1.) However, we wish to emphasize that we have made no attempt in the present paper to produce such a sequential ranking procedure, but rather leave that as a topic for future research.

One would naturally be very much interested in the efficiency (measured in terms of expected number of cycles to terminate the tournament) of our sequential identification procedure of Section 3.2 relative to that of a

single-stage identification procedure which guarantees (3.1) and (3.2). (Note that the determination of the $\hat{c}$ of (2.7) does not involve knowledge of the value of the nuisance parameter θ, while the use of the sequential identification procedure of Section 3.2 does involve such knowledge.) However, we remind the reader that the sequential identification procedure of Section 3.2 which uses the $\hat{P}^*$ of (3.7) is very conservative for $0 < \lambda < 1/3$ because of the crude inequalities (3.9) and (3.10) used in deriving it, i.e., the procedure will actually achieve probabilities of a correct decision under Ω_i^o $(i = 0,1,2,3)$ which are substantially higher than the specified values P_0^* and P_1^*. Thus it may be somewhat misleading to compare this procedure with a single-stage identification procedure which is set up to guarantee the same probability requirements (3.1) and (3.2), unless some adjustment is made for "excess." (See [2], Section 3.7(d) and (e).)

5. Relation between the Sobel-Weiss paper and the present paper

The Sobel-Weiss paper [4] describes selection procedures for tournaments with 2 or 3 players. Their formulation of the problem for tournaments with 3 players is analogous to ours except that they never permit the decision d_0, and hence they consider only the probability requirements R_1 of (2.2) and R_1^o of (3.2). Ideally they would like all of their sequential procedures for tournaments with 3 players to guarantee our probability requirement R_1 of (2.2) (their requirement (1.1)). However, at this point in the development of the theory, none of their procedures has been proved to do so. They have proved that their procedure R_I does so if the value of the nuisance parameter $\theta = p_{jk}$ in our (2.2) is known. Their procedure R_E is shown to guarantee our probability requirement R_1^o of (3.2). Their procedure R_S is intended to guarantee R_1^o of (3.2), it is studied using Monte Carlo sampling, but no analytic results are obtained.

The Sobel-Weiss procedures R_{DL}, R_{DD}, R_{DDR}, R_{IC}, and R_{BKS} guarantee our R_1^o of (3.2); they use the same stopping rule and terminal decision rule as does our sequential identification procedure of Section 3.2 when we let $\lambda = 1/3$ in our procedure, but they use different sampling rules. (Their procedure R_{BKS} is identical to our procedure when we let $\lambda = 1/3$.) Their sampling rules could be used with our procedure of Section 3.2 for arbitrary λ $(0 \leq \lambda \leq 1/3)$ to guarantee (3.1) and (3.2) simply by redefining the L_m's of (3.3) to read $L^{(ab,cd,ef)}_{(m_{ab},m_{cd},m_{ef})}$ since in general we will have $s^{(ab)}_{m_{ab}}$, $s^{(cd)}_{m_{cd}}$, $s^{(ef)}_{m_{ef}}$ with $m_{ab} \neq m_{cd} \neq m_{ef}$. The improvements in $\bar{N}$ values (see the tables in [5] summarizing the results of Monte Carlo sampling experiments) obtained using their sampling rules may very well carry over to our procedure for arbitrary λ $(0 \leq \lambda \leq 1/3)$, and this possibility is certainly worth investigating further.

It should be pointed out that a major drawback of the Sobel-Weiss 3-decision formulation is that $E\{n\}$ will become arbitrarily large for their five procedures R_{DL}, R_{DD}, R_{DDR}, R_{IC}, and R_{BKS} if $\min\{p_{12}, p_{23}, p_{31}\} \to 1$ or $\min\{p_{21}, p_{13}, p_{32}\} \to 1$ (since this means that d_0 is the true State of Nature). However, our procedure will tend to terminate early in either of these situations, and make the decision d_0.

6. Directions of future research

It would be desirable to continue the present investigation to find sequential ranking procedures which can be proved analytically to guarantee the probability requirements (2.1) and (2.2), and which employ some of the sampling rules (or variations thereof) studied by Sobel and Weiss.

Research should be devoted to finding improved values of the stopping constant $\hat{P}*$ of (3.7) in order to cut down on the overprotection. It would also be interesting to generalize this approach to tournaments involving more than 3 players.

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13. ABSTRACT

M. Sobel and G. Weiss have proposed sequential selection (identification and ranking) procedures for 3-player tournaments. In their formulation of the problem they permit only one of the three terminal decisions: "Player i is 'best.'" (i=1,2,3)

In the formulation described in the present paper, a fourth decision is also permitted, namely, "There is no clear-cut 'best' player in the tournament"--to cope with those situations for which the possibility exists that (say) Player 1 is better than Player 2, Player 2 is better than Player 3, and Player 3 is better than Player 1. For this 4-decision formulation, a single-stage <u>ranking</u> procedure is proposed which guarantees that certain requirements on the probability of a correct selection will be achieved (no matter which one of the four States of Nature is the true one); for this same formulation, a sequential <u>identification</u> procedure is proposed which guarantees that certain more restrictive requirements on the probability of a correct selection will be achieved. Our sequential identification procedure can be regarded as a generalization of one of the sequential procedures proposed by Sobel and Weiss.

The relationship between certain aspects of the Sobel-Weiss paper and the present paper is discussed in some detail. Directions of future research are proposed.

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